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A new look at unitary–antiunitary representations of groups and their construction

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Abstract. One starts with a representation of a given abstract group $G = H + sH$ (where H is its subgroup of index 2, and s is a coset representative) by a group of unitary and antiunitary operators in the state space of a quantum system. Co-representation theory maps further these operators into matrices, which are linear operators in the space C^n of number columns, but it fails to preserve homomorphism. An equivalent theory in terms of unitary matrices and antimatrices (antilinear operators in C^n) which is based on isomorphism with the mentioned group of operators is presented. The connections (subduction on the one side and $*$-induction on the other) between the set of all unitary irreducible matrix representations of H and the set of all unitary irreducible matrix–antimatrix representations of G are studied. Finally, a simple construction of the latter from the former is given.

1. Introduction

Quantum systems which contain an odd number of fermions and have a Hamiltonian with time-reversal symmetry display the so called Kramers' degeneracy (even multiplicity) of their energy levels. Kramers' discovery (Kramers 1930) of this fact inspired Wigner to invent the theory of co-representations (Wigner 1932, 1959, Jansen and Boon 1967). It solved the problem of additional degeneracy due to antilinear symmetry operators in the most general case, and it found application especially in the theory of magnetic groups (Bradley and Cracknell 1972).

We deal with an *abstract group* $G = H + sH$ (here and throughout '+', when it is between two disjoint sets, denotes their union), where H is a subgroup of G of index 2 and s is an arbitrary but fixed coset representative. It is desirable in treating some symmetry problems in quantum mechanics to have a *homomorphic mapping* of G onto a group $\hat{D}_{(a)}(G)$:

$$G \to \hat{D}_{(a)}(G) \equiv \hat{D}(H) + \hat{D}_a(s)\hat{D}(H), \tag{1}$$

where $\hat{D}(H) \equiv \{\hat{D}(h) | h \in H\}$ is a *group of unitary operators* acting in a unitary state space V of a given quantum system, and $\hat{D}_a(s)$ is an *antilinear unitary* (in short: antiunitary) *operator* in V, e.g. the time-reversal operator (Messiah 1962). We write (a) as an index in $\hat{D}_{(a)}(G)$ to indicate that it contains both linear operators $\hat{D}(h)$ and antilinear ones like $\hat{D}_a(s)$, indexed by a without brackets.

In co-representation theory one maps the entire $\hat{D}_{(a)}(G)$ onto a set of unitary matrices, which are in their turn linear operators in the space C^n of number columns. Obviously, the set of matrices which makes up a co-representation of G is *not* a group itself, and, clearly, there is *no homomorphism* of G onto this set.

It is our first aim to show that the elements from the coset $\hat{D}_a(s)\hat{D}(H)$ can be represented by antilinear unitary operators in C^n, and, what is more, we further establish an *isomorphism* of the group of operators $\hat{D}_{(a)}(G)$ onto a *group of unitary matrices and antimatrices* (the latter are defined below). Combining homomorphism (1) with this isomorphism, one sees that the group G is *homomorphically* mapped onto a group of unitary matrices and antimatrices, and so the standard apparatus of group representation theory can be adapted and utilised.

Our second aim is to develop a general and practical method of construction of all the finite-dimensional *unitary irreducible matrix–antimatrix representations* (UMAM irreps) of any given group G assuming that all the finite-dimensional *unitary irreducible matrix representations* (UM irreps) of its invariant subgroup H are known. To this purpose we introduce $*$-induction, the analogue of induction, well known in the theory of linear representations (Jansen and Boon 1967, p 133). For the case when the $*$-induced UMAM representation (UMAM rep) is reducible, we give a shortcut: a direct construction of the UMAM irreps. For those groups H which have no other irreps than finite-dimensional unitary ones (such are finite groups and infinite compact ones), it will turn out that all the UMAM irreps of G are found in this way.

In a recent paper van den Broek (1979) solves a problem which is more ambitious, because he starts with a subgroup of G of an index larger than or equal to two, and he uses the method of generalised induction (Shaw and Lever 1974). This approach seems to be inspired by the well known Clifford–Frobenius solution (Clifford 1937) for the analogous problem in linear representation theory. However, there is an essential difference between a group of unitary operators and a group consisting of unitary and antiunitary ones. Whereas in the latter the index-2 subgroup of all unitary operators plays an obviously exceptional role, in the former there is nothing to distinguish between various invariant subgroups. Thus, in the linear case one naturally wants to have a method of construction of UM irreps of G starting from an invariant subgroup of index n. For construction of UMAM irreps of G it is most natural to confine oneself to $n=2$.

Whenever appropriate, we compare the method of construction of UMAM irreps of G with the analogous procedure for UM irreps.

2. Unitary matrix–antimatrix representations

2.1. *Antimatrices*

An antilinear operator $\hat{A}_a$ in a finite-dimensional unitary vector space V_n is characterised by

$$\hat{A}_a(\alpha x+\beta y)=\alpha^*\hat{A}_a x+\beta^*\hat{A}_a y, \qquad \forall x, y \in V_n, \qquad \alpha, \beta \in C, \tag{2}$$

where the asterisk denotes complex conjugation, and n is the dimension of V_n. The antilinear operator $\hat{A}_a$ is unitary iff

$$(\hat{A}_a x, \hat{A}_a y)=(x, y)^*, \qquad \forall x, y \in V_n, \tag{3a}$$

which leads to the equivalent condition

$$\hat{A}_a^\dagger = \hat{A}_a^{-1}. \tag{3b}$$

We call the antilinear operators in the space C^n of number columns *antimatrices*, because the matrices themselves are linear operators in C^n. In a previous paper (Herbut and Vujičić 1967) we used the longer term 'antilinear matrices', which we abandon now because it might be misunderstood as indicating a special class of matrix.

The antimatrices are very simple to deal with owing to the following facts:

(i) The natural basis $\{(1\ 0\ldots 0)^t, (0\ 1\ 0\ldots 0)^t, \ldots, (0\ldots 0\ 1)^t\}$ (where t denotes transposition) in C^n defines a unique antimatrix K_0 by the requirement that K_0 leaves each vector in this basis invariant. It has the properties:

$$K_0^{-1} = K_0^\dagger = K_0 \tag{4}$$

($K_0^\dagger$ denotes the adjoint), i.e. the antimatrix K_0 is unitary, Hermitian, and by consequence an involution (i.e. $K_0^2 = I$, where I is the unit $n \times n$ matrix); and

$$K_0 A = A^* K_0 \tag{5}$$

for any matrix A.

(ii) Any antimatrix A_a can be written in the form AK_0, where $A \equiv A_a K_0$ is a matrix called the *matrix factor* of A_a. In particular, an antimatrix A_a is unitary iff its matrix factor A is unitary.

2.2. *Homomorphism*

For $\hat{D}_{(a)}(G)$ (cf (1)) a UMAM rep is obtained by choosing an orthonormal basis $\{e_1, e_2, \ldots, e_n\} \subset V_n$ and evaluating the matrix elements of the unitary linear and antilinear operators:

$$D_{ij}(h) = (e_i, \hat{D}(h)e_j), \qquad \forall h \in H, \tag{6}$$

$$D_{ij}(s) = (e_i, \hat{D}_a(s)e_j), \tag{7}$$

where $i, j = 1, 2, \ldots, n$. Thus one maps

$$\left.\begin{array}{l} \hat{D}(h) \to D(h) \qquad (8a) \\ \hat{D}_a(s)\hat{D}(h) \to D_a(s)D(h) \qquad (8b) \end{array}\right\} \forall h \in H$$

where

$$D_a(s) = D(s)K_0 \tag{9}$$

is an antimatrix.

In this way one obtains the UMAM group

$$D_{(a)}(G) \equiv D(H) + D_a(s)D(H), \tag{10}$$

where $D(H) \equiv \{D(h) | h \in H\}$. $D_{(a)}(G)$ is an *isomorphic* image of $\hat{D}_{(a)}(G)$, and consequently a *homomorphic* image of G:

$$\left.\begin{array}{l} D(h)D(h') = D(hh'), \\ D_a(sh)D(h') = D_a(shh'), \\ D(h)D_a(sh') = D_a(hsh'), \\ D_a(sh)D_a(sh') = D(shsh'), \end{array}\right\} \forall h, h' \in H. \tag{11a–d}$$

A word of warning seems to be in place: when the antimatrices in (11) are factorised into their matrix factors and K_0, homomorphism does not apply in general to the matrix factors, e.g. $D(shh') \neq D(sh)D(h')$ (unless $D(h')$ is real, as seen from (11*b*)). Actually, (11*a–d*) only in terms of the matrices $D(h)$ and the matrix factors $D(sh')$ are the known *defining relations of a co-representation* (Jansen and Boon 1967, p 171).

Lemma 1. Let $D(H)$ be a given UM rep of the subgroup H of G. A unitary antimatrix $D_a(s)$ defines the coset $D_a(s)D(H)$, and thus completes $D(H)$ into a UMAM rep $D_{(a)}(G)$ of G, iff

$$D_a^{-1}(s)D(h)D_a(s) = D(s^{-1}hs), \qquad \forall h \in H, \tag{12a}$$

$$D_a^2(s) = D(s^2). \tag{12b}$$

Proof. Equations (12) obviously follow from (11). That the former also imply the latter is a consequence of: $hsh' = s(s^{-1}hs)h'$, and $shsh' = s^2(s^{-1}hs)h'$. QED

Since we shall give a comparative presentation of the construction of the UMAM and the UM irreps of G, it is noteworthy that an analogous lemma (where $D_a(s)$ is replaced by a unitary matrix $D(s)$) holds in the case of UM reps.

2.3. *Irreducibility and equivalence*

Defining the irreducibility of linear–antilinear reps of G in full analogy with the linear case, one can formulate the following criterion for it (it is the linear–antilinear analogue of the First Schur lemma—Jansen and Boon 1967, p 99):

A linear–antilinear rep of G is *irreducible* iff there is no other Hermitian operator but λI, $\lambda \in R$, that commutes with all linear and antilinear operators of the rep (for a proof see Theorem II in Dimmock 1963).

By definition two UMAM reps of G are equivalent if there exists a nonsingular matrix T making one the similarity transform of the other, i.e.

$$D'_{(a)}(g) = TD_{(a)}(g)T^{-1}, \qquad \forall g \in G. \tag{13}$$

If the UMAM reps $D_{(a)}(G)$ and $D'_{(a)}(G)$ are irreducible, T can be chosen to be unitary. Namely, adjoining (13) and making use of (3*b*) and of the homomorphism, one concludes that the Hermitian matrix $TT^\dagger$ commutes with each element of the UMAM irrep $D_{(a)}(G)$. Hence the UMAM analogue of the Schur lemma gives $TT^\dagger = \lambda I$, $\lambda \in R$, so that $U \equiv \lambda^{-1/2}T$ is unitary.

Relation (13) for the matrix factors of unitary antimatrices then becomes

$$D'_a(sh) = D'(sh)K_0 = UD(sh)K_0U^\dagger = UD(sh)U^t K_0,$$

i.e.

$$D'(sh) = UD(sh)U^t, \qquad \forall h \in H. \tag{14}$$

This means that the matrix factors of the unitary antimatrices transform by so called *unitary congruence*. (As to the canonical form under unitary congruence, which is the analogue of the diagonal form under unitary similarity, see Vujičić *et al* 1972 and Herbut *et al* 1974.)

If in a given UMAM rep $D_{(a)}(G)$ (cf (10)) of $G = H + sH$ one confines oneself to H, one obtains the *subduced* UM rep of H denoted by $D_{(a)}(G)\downarrow H$.

Clearly, if two UMAM reps of G are equivalent, so are the subduced UM reps of H. One wonders if the converse is also true. An equivalent, but simpler form of this question is: if $D(H)$ is any given UM rep of H, can there exist two inequivalent UMAM reps of G, both subducing into the same $D(H)$?

The following theorem on equivalence of UMAM reps of G answers this question.

Theorem 1. (*The Basic Theorem*). Two UMAM reps of G are equivalent iff the subduced UM reps of H are equivalent.

A proof for the analogous theorem in co-representation theory is given in Jansen and Boon (1967, p 174). (If one makes use of homomorphism, valid for a UMAM rep and not for a co-representation, this proof becomes much more natural.)

Remark 1. In the case of UM reps one has no analogue of the Basic Theorem.

Remark 2. UMAM reps have no invariant trace for their characterisation and easy manipulation. This is due to the fact that the matrix factors of unitary antimatrices transform by unitary congruence (cf (14)), and not by similarity transformation. The Basic Theorem can be viewed as a natural compensation for this shortcoming: *two UMAM reps are equivalent* iff *the characters of the subduced UM reps of H are equal.*

3. Connections between irreps of G and H

It is the aim of this section to study the *natural connections* between the UM and UMAM irreps of G and the UM irreps of H.

3.1. *Classes of irreps*

Let $D(G) \equiv D(H) + D(s)D(H)$ be a UM rep of G. $\tilde{D}(G) \equiv D(H) + [-D(s)]D(H)$ is called its *associate* rep.

It is easy to show that $D(G) \to \tilde{D}(G)$ maps one UM rep onto another, and one UM irrep onto another. This map is an involution, and consequently it breaks up the set of all inequivalent irreps of G into two-element and one-element classes. The former are called pairs of associates, and the latter self-associates. $\{\chi(g)|g \in G\}$ and $\{\tilde{\chi}(g)|g \in G\}$ are the characters of a pair of associates iff $\chi(h) = \tilde{\chi}(h)$ and $\chi(sh) = -\tilde{\chi}(sh)$, $\forall h \in H$; $\{\chi(g)|g \in G\}$ is the character of a self-associate iff $\chi(sh) = 0$, $\forall h \in H$.

Analogously, by $D(h) \to D(h)$, $\forall h \in H$, and $D_a(s) \to -D_a(s)$ one can define also the mapping of any UMAM rep of G onto its associate: $D_{(a)}(G) \to \tilde{D}_{(a)}(G)$, but they both belong to one self-associate equivalence class because $-D_a(s) = iD_a(s)(-i)$. Therefore, this map does not give any classification of the UMAM irreps of G in contrast with the case of UM irreps, and consequently there will be considerable differences in the mentioned connections (cf Theorems 2 and 3).

The map $D(h) \to \bar{D}(h) \equiv D(s^{-1}hs)$, $\forall h \in H$, is called *s-conjugation* of $D(H)$. The s-conjugations for all choices of s from the coset give one and the same map of the set of all inequivalent UM irreps of H onto itself (Jansen and Boon 1967, p 142). Owing to the involutionary property of this map, this set is broken up into one-element and two-element classes, the so called orbits. A UM irrep of a one-element orbit is called self-s-conjugate or 1st kind irrep, and a UM irrep of a two-element orbit is called non-self-s-conjugate or 2nd kind irrep. $\{\chi(h)|h \in H\}$ is the character of a 1st kind irrep iff $\bar{\chi}(h) \equiv \chi(s^{-1}hs) = \chi(h), \forall h \in H$; $\{\chi(h)|h \in H\}$ and $\{\bar{\chi}(h)|h \in H\}$ are the characters of a pair of mutually s-conjugate irreps iff there exists $h' \in H$, such that $\chi(h') \neq \bar{\chi}(h')$.

The mapping $D(h) \to \bar{D}^*(h) \equiv D^*(s^{-1}hs), \forall h \in H$, we call *complex-s-conjugation*, or by symbol $*$-s-conjugation of $D(H)$. It is easy to see that again, irrespective of the choice of s from the coset, one has one and the same map of the set of all inequivalent UM irreps of H onto itself giving a decomposition into one-element and two-element $*$-*orbits* (which are, in general, different from the orbits). The elements of the former are characterised by $\bar{\chi}^*(h) \equiv \chi^*(s^{-1}hs) = \chi(h)$, $\forall h \in H$, and are called self-$*$-s-conjugate irreps. Any UM irreps of H belong to a two-element $*$-orbit iff there exists $h' \in H$ such that $\bar{\chi}^*(h') \neq \chi(h')$.

3.2. Subduction

Lemma 2. Every UMAM rep of G, $D_{(a)}(G)$, when subduced, gives a self-$*$-s-conjugate UM rep of H.

Proof. Let $\chi(h)$ be the character of $D_{(a)}(G)\downarrow H$. Then the homomorphism gives $D(s^{-1}hs) = K_0 D^{-1}(s) D(h) D(s) K_0$, $\forall h \in H$. Multiplying from the left and from the right by K_0, one obtains $\chi^*(s^{-1}hs) = \chi(h)$, $\forall h \in H$. QED

Remark 3. An analogous lemma is valid in the case of UM reps of G.

Now we can formulate the first natural connection.

Theorem 2. (*Theorem on subduction*)

(a) Let $d(G)$ be any UM irrep of G. When subducing it to H one of the following *two* cases will occur:

(1) $d(G)\downarrow H$ is *irreducible* iff $d(G)$ is non-self-associate; then the obtained $d(H) \equiv d(G)\downarrow H$ is self-s-conjugate, and both $d(G)$ and $\tilde{d}(G)$ subduce into the same $d(H)$ up to equivalence.

(2) $d(G)\downarrow H$ is *reducible* iff $d(G)$ is self-associate; then $d(G)\downarrow H$ reduces into a pair of inequivalent mutually s-conjugate irreps of H.

(b) Let $d_{(a)}(G)$ be any UMAM irrep of G. When subducing it to H one of the following *three* mutually exclusive possibilities will occur:

(1*) The subduced UM rep $d_{(a)}(G)\downarrow H$ is *irreducible*; then $d(H) \equiv d_{(a)}(G)\downarrow H$ is self-$*$-s-conjugate.

(2*) The UM rep $d_{(a)}(G)\downarrow H = D(H)$ is *reducible*; then it reduces into two UM irreps of H which are mutually $*$-s-conjugate: $D(h) \sim d(h) \dot{+} \bar{d}^*(h)$, $\forall h \in H$. There are *two* subcases:

(2*a) The UM irreps $d(H)$ and $\bar{d}^*(H)$ are equivalent.

(2*b) They are inequivalent.

The results of Theorem 2 are not new (Jansen and Boon 1967, pp 162 and 176). They are presented here because they are inseparably tied to the second natural connection (Theorem 3).

Remark 4. Part (b) of the Theorem on subduction has, as a by-product, a classification (1*, 2*a, and 2*b kind) of the UM irreps of H. In the following we give three more classifications, and all of them will be seen to amount to the same (see table 2). Owing to the Basic Theorem, with this *the* UMAM *irreps* $d_{(a)}(G)$ *are classified accordingly.*

3.3. *Induction and ∗-induction*

Lemma 3. Let $\{d(h)|h \in H\}$ be any given UM irrep of H. Then defining

$$D(h) \equiv \begin{pmatrix} d(h) & 0 \\ 0 & \bar{d}^*(h) \end{pmatrix}, \qquad \forall h \in H; \qquad D_a(s) \equiv \begin{pmatrix} 0 & d(s^2) \\ I & 0 \end{pmatrix} K_0, \qquad (15a, b)$$

one obtains a UMAM rep of G: $D_{(a)}(G) \equiv D(H) + D_a(s)D(H)$.

Proof is immediately obtained by checking via ($12a$) and ($12b$) of Lemma 1.

We call this new construction, i.e. ($15a, b$), *∗-induction*. The ∗-induced UMAM rep of G we denote by $d(H) \overset{*}{\uparrow} G$.

As is known from the literature (Jansen and Boon 1967, p 133), the construction analogous to ($15a, b$), but without ∗ and K_0, gives a UM rep of G: $d(H)\overset{*}{\uparrow}G$, and this construction is called *induction*.

The aim of the next theorem is to connect the UMAM irreps of G with the UM irreps of H by using the ∗-induction procedure (and by contrasting it with the known connection for UM irreps of G). From the point of view of construction of UMAM irreps of G it will turn out that ∗-induction is not practical in all cases, and therefore it will be supplemented by a more direct evaluation in Theorem 4.

Theorem 3. (Theorem on induction and on ∗-induction)

(a1) The induced UM rep, $d(H)\uparrow G = D(G)$, is *reducible* iff $d(H)$ is a self-s-conjugate UM irrep of H, i.e. iff there exists a unitary matrix Z such that $\bar{d}(h) = Z^{-1}d(h)Z, \forall h \in H$; then Z can be chosen so that $Z^2 = d(s^2)$ and $D(G)$ reduces into two inequivalent mutually associate UM irreps of G:

$$d(H) + Zd(H) \text{ and } d(H) + (-Z)d(H) \text{(up to equivalence).}$$

(2) The induced UM rep of G is *irreducible* iff $d(H)$ is a non-self-s-conjugate UM irrep of H; then $d(G) \equiv d(H)\overset{*}{\uparrow}G$ is self-associate and $d(H)$ and $\bar{d}(H)$ induce equivalent UM irreps of G.

(b1*) The ∗-induced UMAM rep of G is *reducible* iff there exists a unitary matrix Z such that $\bar{d}^*(h) = Z^{-1}d(h)Z$, $\forall h \in H$, and $ZZ^* = d(s^2)$; then $D_{(a)}(G) \equiv d(H) \overset{*}{\uparrow} G$ reduces into two UMAM irreps of G, both equivalent to $d_{(a)}(G) \equiv d(H) + ZK_0d(H)$.

(2*) When $d(H) \overset{*}{\uparrow} G$ is *irreducible*, then we write it as $d_{(a)}(G)$, and *two* subcases should be distinguished according to whether $d(H)$ and $\bar{d}^*(H)$ in the ∗-induction formula (15a) are equivalent or not:

(2*a) One has a UMAM irrep $d_{(a)}(G)$ and $d(H) \sim \bar{d}^*(H)$ iff there exists a unitary matrix Z such that $\bar{d}^*(h) = Z^{-1}d(h)Z, \forall h \in H$, and $ZZ^* = -d(s^2)$.

(2*b) One has a UMAM irrep $d_{(a)}(G)$ and $d(H) \nsim \bar{d}^*(H)$; a necessary and sufficient condition for this is the inequivalence of $d(H)$ and $\bar{d}^*(H)$ by itself.

Proof. Part (a) of the Theorem is known (Jansen and Boon 1967, p 162), but we have adapted it to be maximally analogous to the UMAM case. To contrast the arguments in the UM and the UMAM cases we give a proof of part (a) in the Appendix.

To prove part (b) of the Theorem, let us start with (2*b), which is simplest to prove. Here $\bar{d}^*(H) \nsim d(H)$ entails irreducibility of $d(H) \overset{*}{\uparrow} G$ because otherwise subduction of the latter (after its reduction into irreducible components) would result (due to Lemma 2) in at least two self-∗-s-conjugate UM reps of H. This would be in contradiction with the fact that $d(H) \dot{+} \bar{d}^*(H)$ is the only self-∗-s-conjugate UM rep available.

The common property of the (1*) and the (2*a) cases is

$$\bar{d}^*(h) = Z^\dagger d(h) Z, \qquad \forall h \in H, \tag{16}$$

where Z is a unitary matrix defined herewith uniquely up to an arbitrary phase factor. The corresponding UMAM reps $d(H)\overset{*}{\uparrow}G$ differ by being reducible or irreducible. According to the linear–antilinear analogue of Schur's lemma (cf § 2) the former commutes with a non-trivial Hermitian matrix, whereas the latter does not. To find necessary and sufficient conditions for the case when such a non-trivial matrix exists, let us write a general Hermitian matrix in block-matrix form (the size of the submatrices is as in (15*a*, *b*)):

$$\begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}, \qquad \text{where } A^\dagger = A, D^\dagger = D.$$

It commutes with (15*a*) iff $A = \alpha I$, $D = \delta I$, $B = \beta Z$, α, $\delta \in R$, $\beta \in C$. Further, it also commutes with (15*b*) iff $\alpha = \delta$ and either $\beta = 0$ or $\beta \neq 0$ and $ZZ^* = d(s^2)$. Hence, $d(H)\overset{*}{\uparrow}G$ is reducible iff

$$ZZ^* = d(s^2). \tag{17}$$

It is straightforward to check that the unitary matrix

$$T \equiv (2)^{-1/2}\begin{pmatrix} \mathrm{i}I & -\mathrm{i}Z \\ I & Z \end{pmatrix}$$

transforms by similarity $d(H)\overset{*}{\uparrow}G$ into $d_{(\mathrm{a})}(G) \dot{+} d_{(\mathrm{a})}(G)$, where $d_{(\mathrm{a})}(G) \equiv d(H) + ZK_0 d(H)$.

As suggested by (17), there is a general relation between Z and $d(s^2)$ derived as follows: inserting $K_0^2 = I$, replacing h by shs^{-1} and taking the similarity transform by Z of equation (16), one obtains

$$(ZK_0)d(h)(ZK_0)^{-1} = d(shs^{-1}) \tag{18}$$

leading to $(ZK_0)^2 d(h)(ZK_0)^{-2} = d(s^2)d(h)d^{-1}(s^2)$, which implies (through Schur's lemma for UM irreps)

$$d^{-1}(s^2)(ZK_0)^2 = \mathrm{e}^{\mathrm{i}\varphi} I,$$

i.e.

$$ZZ^* = \mathrm{e}^{\mathrm{i}\varphi} d(s^2). \tag{19}$$

It is important to note that $\mathrm{e}^{\mathrm{i}\varphi}$ does not depend on the phase factor in the choice of Z, so that it is a characteristic property of the UM irrep $d(H)$ (being one and the same for all the UM irreps equivalent to it).

Applying $(ZK_0)\ldots(ZK_0)^{-1}$ to (19) and taking into account (18) one concludes that $\mathrm{e}^{\mathrm{i}\varphi}$ has to be real, i.e. ± 1. Since the + sign corresponds to the reducible case, the remaining possibility, i.e. the − sign, corresponds to the (2*a) case. QED

Remark 5. The classification of the UM irreps of H used in the Theorem on $*$-induction, i.e. (1*), (2*a) and (2*b), is the same as the one that appeared in the Theorem on subduction. This is obvious from the fact that induction and $*$-induction on the one hand and subduction on the other (as procedures which connect UM and UMAM irreps of G with UM irreps of H) are in a broader sense inverse to each other. (For the direct inversion of subduction see the Conclusions.)

In each of the following tables we summarise the different definitions of the same classification of the UM irreps of H connected with the UM and the UMAM irreps of G respectively. In the latter we add the character test known in the literature (Dimmock 1963). In co-representation theory one uses the terms first, second, and third kind instead of (1*), (2*a) and (2*b) respectively (Jansen and Boon 1967, p 181).

Table 1. Types of UM irreps of H (relevant for construction of UM irreps of G). The three alternative definitions of the same classification (given for comparison with table 2).

Origin of the definition \ Types of UM irreps of H	1	2
Induction	giving $d(H)\uparrow G$ which is reducible	giving $d(H)\uparrow G$ which is irreducible
Subduction	obtainable as $d(G)\downarrow H$ which is irreducible	obtainable through $d(G)\downarrow H$ which is reducible into $d(H)\dot{+}\bar{d}(H)$
Via orbits	one-element orbit (i.e. $d(H)\sim\bar{d}(H)$)	two-element orbit (i.e. $d(H)\nsim\bar{d}(H)$)

Table 2. Types of UMAM irreps of G. The four alternative definitions of the same classification of the UM irreps of H, which give the corresponding classification of the UMAM irreps of G due to Theorem 1.

Origin of the definition \ Types of UM irreps of H	(1*)	(2*a)	(2*b)
$*$-induction	giving $d(H)\overset{*}{\uparrow}G$ which is reducible (into twice the UMAM irrep $d(H)+ZK_0d(H)$)	giving $d(H)\overset{*}{\uparrow}G$ which is irreducible and $d(H)\sim\bar{d}^*(H)$	$d(H)\nsim\bar{d}^*(H)$
Subduction	obtainable as $d_{(a)}(G)\downarrow H$ which is irreducible	obtainable through $d_{(a)}(G)\downarrow H$ which is reducible into $d(H)+\bar{d}^*(H)$, and $d(H)\sim\bar{d}^*(H)$	$d(H)\nsim\bar{d}^*(H)$
Via $*$-orbits	one-element $*$-orbit, i.e. there exists Z, $\bar{d}^*(h)=Z^{-1}d(h)Z$, $\forall h\in H$ and $ZZ^*=d(s^2)$	$ZZ^*=-d(s^2)$	two-element $*$-orbit, i.e. $d(H)\nsim\bar{d}^*(H)$
Character test	$X=1$	$X=-1$	$X=0$

where $X\equiv(1/|H|)\sum_{h\in H}\chi[(sh)^2]$

The $*$-induction method achieves all finite-dimensional UMAM irreps of G when all the finite-dimensional UM irreps of H are made use of. This is a direct consequence of the Basic Theorem and the fact that $*$-induction and subduction are essentially inverse to each other. (The corresponding conclusion is reached also in co-representation theory, Wigner 1959, p 344.) In the case of UM irreps of G the role of the Basic Theorem is taken over by a completely different argument (see Zak 1960).

4. Evaluation of the matrix Z

4.1. Construction

In the cases of the (1) and (1*) kind UM irreps of H, induction and $*$-induction give reducible UM and UMAM reps of G. For their reduced forms the respective Z matrices are indispensable, since, as proved, these forms are: $d(H)+(\pm Z)d(H)$ in the UM case, and $d(H)+ZK_0d(H)$ in the UMAM case. In the following theorem we treat the UM and UMAM cases simultaneously (the latter in square parentheses).

Theorem 4. (*Theorem on the evaluation of Z*). Let $d(H)$ be a given UM irrep of H of the (1) [1*] kind. The unitary matrix Z satisfying

$$Z^{\dagger}d(h)Z=d^{[*]}(s^{-1}hs), \qquad \forall h\in H, \tag{20}$$

can be evaluated as follows.

Defining the $n\times n$ matrices

$$P_i\equiv(n/|H|)\sum_{h\in H}d_{i1}^{*}(h)d^{[*]}(s^{-1}hs), \qquad i=1,2,\ldots,n, \tag{21}$$

one sees that P_1 is a ray projector. One evaluates its eigenvector x (as a number column), of norm 1, corresponding to the eigenvalue 1. Then the ith row Z_i of Z is found as

$$Z_i=x^{\dagger}P_i^{\dagger}, \qquad i=1,2,\ldots,n. \tag{22}$$

The open phase factor of Z, which is its known indeterminacy, results from the arbitrary phase factor in the choice of the eigenvector x.

Proof. From the orthogonality relations for the matrix elements of $d(H)$, i.e.

$$(n/|H|)\sum_{h\in H}d_{11}^{*}(h)d_{ij}(h)=\delta_{i1}\delta_{1j},$$

it follows that

$$(n/|H|)\sum_{h\in H}d_{11}^{*}(h)d(h)=(1\ 0\ldots 0)^{t}(1\ 0\ldots 0). \tag{23}$$

Since the unitary matrix Z exists, we may apply $Z^{\dagger}\ldots Z$ to (23) and, in view of (20) and (21), one obtains $P_1=Z_1^{\dagger}Z_1$. Putting $x\equiv Z_1^{\dagger}$, one has $P_1=xx^{\dagger}$, implying $P_1x=x$.

Analogously, the orthogonality relations

$$(n/|H|)\sum_{h\in H}d_{i1}^{*}(h)d_{jk}(h)=\delta_{ji}\delta_{1k},$$

understood as the (j,k) element of a matrix equation, after application of $Z^{\dagger}\ldots Z$ go over into $P_i=Z_i^{\dagger}Z_1$. Adjoining one obtains $P_i^{\dagger}=Z_1^{\dagger}Z_i$, giving $Z_1P_i^{\dagger}=Z_i$, because $Z_1Z_1^{\dagger}=x^{\dagger}x=1$. QED

4.2. Practical aspects

To give the practical aspects of the construction of the UMAM irrep of G out of a given UM irrep $d(H)$ of H, we suggest the following procedure.

(1) If $d(H)$ is *odd-dimensional*, then one should compute $\bar{d}^{*}(H)$ and find out if $d(H)$ and $\bar{d}^{*}(H)$ are equivalent or not by comparing their characters. If $d(H)\not\sim\bar{d}^{*}(H)$,

i.e. in the (2^*b) case, one uses the $*$-induction formulae (15*a*, *b*). Otherwise, one makes use of the following lemma.

Lemma 4. *If* $d(H) \sim \bar{d}^*(H)$ *and* $\dim[d(H)]$ *is odd, then the* (1^*) *case is characterised by* $\det[d(s^2)] > 0$, *whereas the* (2^*a) *case occurs iff* $\det[d(s^2)] < 0$.

Proof. Taking the determinant of $ZZ^* = \pm d(s^2)$ (cf table 2) one obtains $\det[d(s^2)] = \pm|\det Z|^2$. QED

Thus, if $\det[d(s^2)] > 0$, one should construct Z as described in Theorem 4, so that

$$d_{(a)}(G) \equiv d(H) + ZK_0 d(H). \tag{24}$$

If $\det[d(s^2)] < 0$, then $d_{(a)}(G)$ is computed via the $*$-induction formulae (15a, b).

(2) If $d(H)$ is *even-dimensional*, then one can choose one of the following two ways:

(a) From the given character table of $d(H)$ one evaluates $X \equiv (1/|H|) \Sigma_{h \in H} \chi[(sh)^2]$, which must be $+1$, -1 or 0. If $X = +1$, one constructs Z as described in Theorem 4 to obtain (24). If $X = -1$ or 0, then $d_{(a)}(G)$ is computed via the $*$-induction formulae (15*a*, *b*).

(b) As in the odd-dimensional case, having computed $\bar{d}^*(H)$, one easily infers whether one is dealing with two $*$-orbits or with one. If the former is true, one utilises (15*a*, *b*). If $d(H) \sim \bar{d}^*(H)$, one may evaluate Z (Theorem 4) and compute $ZZ^* d(s)^{-2}$, which must be $+1$ or -1. In the former case $d_{(a)}(G)$ is given by (24), and in the latter by

$$h \to \begin{pmatrix} d(h) & 0 \\ 0 & d(h) \end{pmatrix}, \forall h \in H, \qquad s \to \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix} K_0. \tag{25}$$

It is easy to ascertain via (12*a*, *b*) that these matrices form a UMAM irrep of G. It is equivalent to the one obtained by (15*a*, *b*) according to the criterion in Remark 4.

5. Conclusions

Our approach rests on two basic ideas, which we believe to be novel: (1) *antimatrices*, and (2) *direct inversion of subduction*, consisting in $*$-induction (Theorem 3) and construction of the matrix Z (Theorem 4):

(1) Clearly, there is no other way to achieve homomorphism $G \simeq \hat{D}_{(a)}(G) \cong D_{(a)}(G)$ ($\simeq$ and $\cong$ denote homomorphism and isomorphism respectively), but to do the natural thing: to utilise antilinear operators in C^n (i.e. antimatrices). We hope to have shown that the gain from the homomorphism $G \simeq D_{(a)}(G)$ outweighs the fact that one is not used to the antimatrices. The more so because the homomorphism enables one to trace the analogies and the differences between the linear and the linear–antilinear representations.

It is noteworthy that the need for linear–antilinear representations is due to quantum mechanical reasons (the first map $G \simeq \hat{D}_{(a)}(G)$). The second map $\hat{D}_{(a)}(G) \cong D_{(a)}(G)$ is nothing but representing $\hat{D}_{(a)}(G)$ in an orthonormal basis in the state space of the quantum system. An important point to note is that representation of $\hat{D}_{(a)}(G)$ is very useful primarily because both the absolute basis and K_0 (defined by the former) are unique and have simple properties as mathematical objects. The antimatrix K_0 does not depend either on the orthonormal basis in the state space or on the element $sh \in G$

which is represented by $D_a(sh) = D(sh)K_0$. Because of this uniqueness, K_0 can be eliminated from $D_{(a)}(G)$, going over to the co-representation. But, as we have tried to show, it pays to keep K_0, because without it one does not have homomorphism and the latter gives valuable conceptual and practical simplifications.

(2) A careful look at table 2 makes it clear that the procedure of subduction is very different in the (1*) case on the one hand, and the (2*a) and (2*b) cases on the other. Hence, *the direct inversion of subduction* is necessarily piecewise: in the (1*) case one has to add the coset $\{D_a(sh)|h \in H\}$, which hinges on the construction of $D_a(s) \equiv ZK_0$; whereas in the (2*) cases it is $*$-induction that achieves the direct inversion of subduction. It seems to us that subduction and its direct inverse are the most natural and the simplest possible connections between the UMAM irreps of G and the UM irreps of H. Therefore, it is best to base the construction of the former on this connection.

It should be pointed out that $*$-induction gives a standard form in the (2*a) case (cf (15a, b)) which is different from that of Wigner, given by the matrix factor in (25) (cf also van den Broek 1979).

It is known that co-representation theory found its most important application in the Shubnikov black-and-white (or magnetic) point and space groups. It might be worthwhile to take a new look at the co-representations of these groups from the point of view of the UMAM irreps. However, our aim is somewhat different. We have constructed the line groups and their irreps (Vujičić *et al* 1977 and Božović *et al* 1978) as the symmetry groups of stereoregular polymer molecules. Work is in progress on constructing the black-and-white line groups and thereafter their UMAM irreps along the lines expounded in this paper.

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Appendix (Proof of the Theorem on Induction)

The induction procedure, which is common to both cases (1) and (2), has two immediate consequences:

(i) $\chi'(h) = \chi(h) + \bar{\chi}(h)$, $\forall h \in H$, where χ and χ' are the characters of $d(H)$ and of $d(H)\uparrow G$ respectively (cf (15a, b)).

(ii) $\chi'(sh) = 0$, $\forall h \in H$ (cf (15b)). As a consequence of (i) and (ii) one obtains: $(1/|G|)\sum_{g \in G} \chi'(g)\chi'^*(g) = (1/2|H|)\sum_{h \in H}[\chi(h) + \bar{\chi}(h)] \times [\chi(h) + \bar{\chi}(h)]^*$, or in terms of scalar products

$$(\chi'|\chi') = (\tfrac{1}{2})(\chi + \bar{\chi}, \chi + \bar{\chi}). \tag{A1}$$

As seen from (A1), $\chi = \bar{\chi}$ implies $(\chi'|\chi') = 2$, whereas $\chi \neq \bar{\chi}$ entails $(\chi'|\chi') = 1$. This means that $d(H)\uparrow G$ is reducible iff $d(H) \sim \bar{d}(H)$, and the former is irreducible iff $d(H) \nsim \bar{d}(H)$.

When $d(H)\uparrow G$ is reducible, then, because of $(\chi'|\chi') = 2$, it is seen to decompose into two inequivalent UM irreps of G:

$$\chi'(g) = \chi'_1(g) + \chi'_2(g), \qquad \forall g \in G.$$

Since $D(H)\uparrow G$ is always self-associate,

$$\chi_1'(g)+\chi_2'(g)=\tilde{\chi}_1'(g)+\tilde{\chi}_2'(g), \qquad \forall g \in G.$$

This requires that either $\chi_1'(g)=\tilde{\chi}_1'(g)$ and $\chi_2'(g)=\tilde{\chi}_2'(g)$, or $\chi_2'(g)=\tilde{\chi}_1'(g)$, because the decomposition of any rep into irreps is unique. The former is impossible because the subduced UM rep of $d(H)\uparrow G$ to H would consist of four UM irreps of H (cf Theorem 2) in contradiction with the linear analogue of (15*a*). Thus, the two inequivalent UM irreps of G are mutually associate.

It is straightforward to see that the similarity transformation by $T\equiv (2)^{-1/2}\begin{pmatrix} I & Z \\ I & -Z\end{pmatrix}$ makes the linear analogue of (15*a*, *b*), i.e.

$$D(h)\equiv\begin{pmatrix} d(h) & 0 \\ 0 & \bar{d}(h)\end{pmatrix}, \qquad D(s)\equiv\begin{pmatrix} 0 & d(s^2) \\ I & 0\end{pmatrix}, \tag{A2}$$

quasidiagonal as stated in the Theorem. For this one needs $\bar{d}(h)=Z^{\dagger}d(h)Z$ (where Z is obviously determined up to an arbitrary phase factor), and $d(s^2)=Z^2$. The latter follows from $Z^2=e^{i\varphi}d(s^2)$, which is obtained by analogy with the UMAM case (cf (19)). However, there is an essential difference between the UM and the UMAM cases, namely, here φ depends on Z and can be made zero by a suitable phase factor in the choice of Z.

QED

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